Energy transfer between external and internal gravity waves

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In a two-layer liquid system non-linear resonant interactions between a pair of external (surface) waves can result in transfer of energy to an internal wave when appropriate resonance conditions are satisfied. This energy transfer is likely to be more powerful than similar transfers between external waves. The shallow water case is discussed in detail.

1. Introduction

In this paper we consider the transfer of energy between internal and external gravity waves by 'primary resonant interaction' (a term to be explained subsequently). It was shown by Phillips (1960) that simple surface gravity waves cannot interact in this way and that it was necessary to consider secondary interactions to understand the process of energy transfer. The secondary interactions are apparently significant in the development of a random field of gravity waves such as is generated by the wind blowing over the ocean. It is shown here that, if the liquid is stratified, there is the possibility of *primary* interactions between internal and external waves. Furthermore, as both ocean and atmosphere *are* stratified, it is likely that primary interactions are of some importance in the study of both. In particular the development of the ocean wave spectrum (considered, for instance, by Benney 1962; Hasselmann 1962, 1963*a*, *b*; Longuet-Higgins 1962; Phillips 1960) may be partly controlled by energy exchanges with internal motions; the internal motions so generated are also of considerable interest in themselves.

The process of resonant interaction is not difficult to understand physically, though the detailed analysis of a particular case generally involves a great deal of algebra which effectively obscures the simple nature of the process. We propose, therefore, before considering gravity-wave interactions, to give a simple description of the process without reference to a specific physical system. For simplicity we confine our attention to systems which, when the governing equations are linearized by perturbation methods, give equations with constant coefficients having solutions expressible as the sum of undamped waves of the form $\sin(\mathbf{m}.\mathbf{r}-\nu t+\epsilon)$. The vector wave-number $\mathbf{m}$ and the frequency ν satisfy a frequency equation

$$F(\mathbf{m},\nu) = 0, \tag{1.1}$$

which is characteristic of the particular set of linear equations. These waves are mutually independent and each conserves its own energy; the phase velocity, energy transmission (group velocity) and dispersion will all be described more or less adequately by this linear theory. If the system is a very simple one, where time is the only independent variable, then the frequency equation is in ν only and has discrete roots which give the frequencies of the normal modes.

We now investigate circumstances under which the non-linear terms can give rise to significant energy transfer between the basic wave solutions. Suppose there are two waves (which we will call primary waves) with vector wave-numbers $\mathbf{m}_1$ and $\mathbf{m}_2$ and frequencies ν_1 and ν_2; these waves are both solutions of the linearized equations and each satifies the frequency equation (1.1). The most important non-linear terms are usually the quadratic ones, though there are systems where the quadratic terms vanish and the important terms are cubic or higher powers; we will not be concerned with these. When two primary waves are present the quadratic terms can be expressed as sum and difference (combination) waves with wave-numbers $\mathbf{m}_3 = \mathbf{m}_1 \pm \mathbf{m}_2$ and frequencies $\nu_3 = \nu_1 \pm \nu_2$; there may also be a 'self-interaction' of each wave, included in the present remarks by putting $\mathbf{m}_1 = \mathbf{m}_2$ and $\nu_1 = \nu_2$, though such an interaction is not of interest here because it does not play a direct part in the energy transfer between internal and external waves.

These combination waves can be considered as forcing waves acting on the linearized system whose response can readily be determined by solving the appropriate equations. It is perhaps preferable, in the first instance, to explain what happens in physical terms without recourse to mathematical details. The combination waves move with phase speeds $\nu_3/|\mathbf{m}_3|$ which are determined by the primary waves. They generate secondary waves of wave-number $\mathbf{m}_3$ that move at phase speeds characteristic of the system, that is, at speeds $\nu_j/|\mathbf{m}_3|$ where the ν_j are the roots of the frequency equation $F(\mathbf{m}_3, \nu) = 0$. In general none of the ν_j will equal ν_3 and the secondary waves will move at different phase speeds from the combination waves. Whereas initially there will be a transfer of energy to the secondary waves, none of these waves will remain in phase with either of the combination waves so that continuous transfer of energy will not take place and the amplitudes of the secondary waves will remain small. The maximum amplitude attained will be greater the longer the wave remains in, or nearly in, correct phase; that is the more nearly ν_3 equals ν_j.

The phenomenon of *primary resonant interaction* occurs when one of the combination waves moves at the same speed as one of the secondary waves (i.e. $\nu_3 = \nu_j$). This particular secondary wave does not now get out of phase with the forcing wave and continuous transfer of energy is possible. This transfer will, of course, be limited by the available energy in the primary waves.† The condition for resonant interaction between the primary waves $(\mathbf{m}_1, \nu_1)$ and $(\mathbf{m}_2, \nu_2)$ is

$$F(\mathbf{m}_1, \nu_1) = F(\mathbf{m}_2, \nu_2) = F(\mathbf{m}_3, \nu_3) = 0, \tag{1.2}$$

and this is also exactly the condition for resonant interaction between the secondary wave and either of the primary waves. The secondary wave therefore

† In reality the energy transfer will also be limited by the finite lengths of the wave trains. Transfer can only take place while the wave trains overlap and the time of overlap is dependent on the *group* velocities in contrast to the condition for resonant interaction which is dependent on the *phase* velocities.

interacts with either primary to transfer energy to the other and we have a triplet of waves which exchange energy comparatively freely. If other energy transfers are ignored we find in some systems that the rate of change of amplitude of each wave is proportional to the product of the amplitudes of the other two (as in the case considered in § 3). The amplitudes of the waves can then be expressed as elliptic functions of time whose period (which is much longer than the periods of the individual waves) depends on the total energy of the three waves.

The preceding remarks will now be illustrated by consideration of a simple second-order system. Suppose that the linearized equation of the system is

$$L(y) = 0, \tag{1.3}$$

where L is a second-order linear differential operator and y is a dependent variable characterising the state of the system. If we put y equal to

$$y_0 \cos(\mathbf{m}.\mathbf{r} - \nu t)$$

in the left-hand side of equation (1.3), we obtain $F(\mathbf{m}, \nu) y_0 \cos(\mathbf{m}.\mathbf{r} - \nu t)$, where $F(\mathbf{m}, \nu)$ is the function that appears in the frequency equation (1.1); clearly if $\mathbf{m}$ and ν satisfy the frequency equation then $y_0 \cos(\mathbf{m}.\mathbf{r} - \nu t)$ is a solution of equation (1.3). When two such primary waves are present the secondary waves are determined by an equation of the form

$$L(y) = P\cos(\mathbf{m}_3.\mathbf{r} - \nu_3 t), \tag{1.4}$$

where the right-hand side is derived from the quadratic non-linear terms, giving rise to a combination wave, and P is proportional to the product of the amplitudes of the primary waves, the constant of proportionality being dependent on the form of the non-linear terms. P is regarded as constant, though it will in fact vary slowly with time as the amplitudes of the primary waves change.

If there are no secondary waves present initially then the appropriate initial conditions for a second-order system are

$$y = 0 \quad \text{and} \quad dy/dt = 0 \quad \text{when} \quad t = 0. \tag{1.5}$$

The general solution of equation (1.4) can be expressed as the sum of a particular integral and a complementary function, the latter in this case being an arbitrary set of 'free' waves, that is waves subject to the condition expressed by (1.1). A particular integral is clearly

$$y = \frac{P}{F(\mathbf{m}_3, \nu_3)} \cos(\mathbf{m}_3.\mathbf{r} - \nu_3 t), \tag{1.6}$$

provided $F(\mathbf{m}_3, \nu_3)$ *is not zero*. In order to satisfy the initial conditions (1.5) we must select from the arbitrary set of free waves two waves, each of wave-number $\mathbf{m}_3$, and of appropriate amplitude. The frequencies of these two free waves are the roots ν_a, ν_b of the equation $F(\mathbf{m}_3, \nu) = 0$ (there are two roots because we are considering a second-order system). We then find that

$$y = \frac{P}{F(\mathbf{m}_3, \nu_3)} \left\{ \cos(\mathbf{m}_3.\mathbf{r} - \nu_3 t) + \left(\frac{\nu_3 - \nu_b}{\nu_b - \nu_a}\right) \cos(\mathbf{m}_3.\mathbf{r} - \nu_a t) + \left(\frac{\nu_3 - \nu_a}{\nu_a - \nu_b}\right) \cos(\mathbf{m}_3.\mathbf{r} - \nu_b t) \right\}. \tag{1.7}$$

The amplitudes of these waves are of the second order of smallness (provided $F(\mathbf{m}_3, \nu_3)$ is not close to zero) if the amplitudes of the primary waves are small.

When primary resonant interaction occurs we have $F(\mathbf{m}_3, \nu_3) = 0$, and the solution can be derived from equation (1.7) by letting ν_3 approach one of the roots of $F(\mathbf{m}_3, \nu) = 0$, ν_a say. We then obtain

$$y = \frac{P}{F'}\left\{t \sin(\mathbf{m}_3 . \mathbf{r} - \nu_3 t) + \frac{\cos(\mathbf{m}_3 . \mathbf{r} - \nu_b t) - \cos(\mathbf{m}_3 . \mathbf{r} - \nu_3 t)}{\nu_3 - \nu_b}\right\}, \tag{1.8}$$

where F' is the value of $\partial F/\partial \nu$ when $\mathbf{m} = \mathbf{m}_3$ and $\nu = \nu_3$. It can readily be verified that this is a solution of equation (1.4) subject to the initial conditions (1.5) and the condition $F(\mathbf{m}_3, \nu_3) = 0$. We see from equation (1.8) that the amplitude of the $(\mathbf{m}_3, \nu_3)$ wave *grows linearly in time*. This is the important case of primary resonant interaction. No matter how small P is, given sufficient time, this wave can grow in amplitude until it becomes significantly large. If we denote the amplitude of the wave $(\mathbf{m}_3, \nu_3)$ by η_3 and assume that P is a slowly varying function of time (i.e. $dP/dt \ll \nu_3 P$, an assumption that is justified *a posteriori* for the case considered in § 3) then we find that

$$d\eta_3/dt = P/F' \tag{1.9}$$

an equation to which we refer subsequently.

A special case of particular interest occurs when $\nu = 0$ is a solution of the frequency equation; 'waves' of the form $\sin(\mathbf{m} . \mathbf{r} + \epsilon)$ are then possible solutions of the linearized equations for all values of $\mathbf{m}$ and ϵ. The phase velocity and group velocity are both zero and the system is quasi-static, its state being changed only by non-linear effects. Furthermore, *all* these waves interact resonantly to transfer energy to other waves of the same type, provided non-linear effects are present, and as a result resonant interactions are not confined to particular triplets and energy exchanges within this set of 'waves' are extremely complicated. An example of this type of behaviour is provided by the turbulent motions of a homogeneous fluid.

A further interesting special case occurs when the phase speed is independent of the wave-number, i.e. when the system is non-dispersive. All waves proceeding in the same direction can then interact resonantly. Such a group of waves can therefore change its form comparatively easily by non-linear interactions but will not easily lose energy to waves moving in any other direction.

2. Conditions for primary resonant interaction in a two-layer liquid system

We now show that primary resonant interaction can occur in a two-layer liquid system with a free upper surface. The frequency equation for such a system (with no stream velocity) is

$$\nu^4(\rho + \rho' \tanh mh \tanh mh') - \nu^2 \rho (\tanh mh + \tanh mh') gm + (\rho - \rho') g^2 m^2 \tanh mh \tanh mh' = 0, \tag{2.1}$$

where the primed quantities refer to the upper layer and the unprimed quantities to the lower, h and h' are the equilibrium depths of the layers (see Lamb's

Hydrodynamics, § 231). Let us, for the moment, confine our attention to waves moving in the x direction (positive or negative). We take ν as positive and the sign of $\mathbf{m}$ then determines the direction of propagation of the wave. The solution curves of equation (2.1) in the (m, ν)-plane are sketched in figure 1. The curves OE_1 and OE_2 represent external waves and the curves OI_1 and OI_2 represent internal waves.

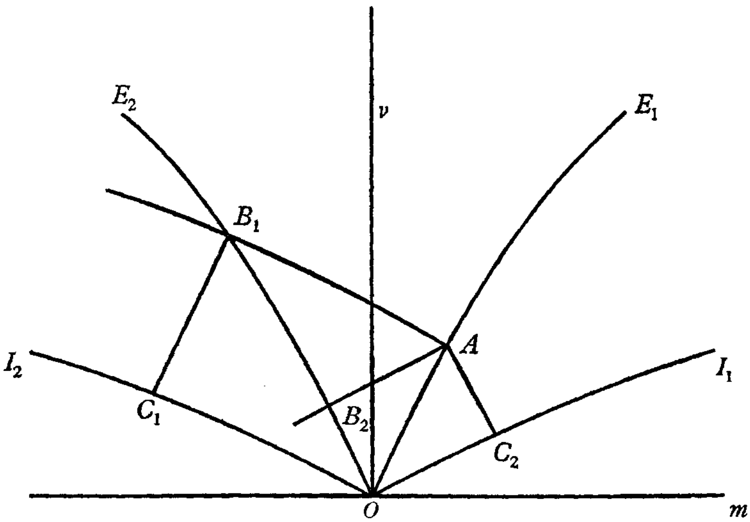

FIGURE 1. Schematic representation of the conditions for resonant interaction in the one-dimensional case. The external wave represented by A belongs to the two resonantly interacting wave triplets represented by A, B_1, C_1 and A, B_2, C_2.

Consider a particular external wave travelling in the positive x direction represented by the point A on the curve OE_1, as indicated in figure 1. If we draw a curve commencing at A, congruent to, and with the same orientation as OI_2, this will intersect OE_2 at just one point, B_1 say. If we denote by C_1 the point OI_2 such that $OC_1 = AB_1$ then the waves represented by the points A, B_1 and C_1 evidently form a resonantly interacting triplet since $\mathbf{OB}_1 = \mathbf{OA} + \mathbf{OC}_1$ and A, B_1 and C_1 are clearly all solution points of equation (2.1). Similarly, if we draw a curve commencing at A congruent to OI_1, but rotated through 180°, this will intersect OE_2 at one point only, B_2 say, with a corresponding point C_2 on OI_1. The points A, B_2 and C_2 form a resonantly interacting triplet in which the roles of A and B are now reversed. There are no other interactions involving A (except for the shallow water case where any two waves of similar type moving in the same direction interact resonantly to produce a secondary of the same type moving in the same direction). These resonantly interacting triplets always involve two external waves moving in the opposite direction to one another and one internal wave.

The solution surfaces of equation (2.1), for waves moving horizontally in any direction, are surfaces of revolution (resembling cones near the origin), obtained by rotating the curves OI and OE about the ν-axis. Let us, as in the one-dimensional case, consider an arbitrary point A on the external wave surface. If we imagine the internal wave cone to be completed by the addition of its image in the **m**-plane and then moved without change of orientation, so that its apex previously at O is now at A; then the curve of intersection between the 'E cone' and the completed 'I cone' gives the waves that interact resonantly with the wave represented by A (see figure 2). For any given direction of propagation, different from that of A, there are just two external waves that interact with A, one with wave-number greater than A and one with wave-number less than A. This is also illustrated in a two-dimensional diagram (figure 3).

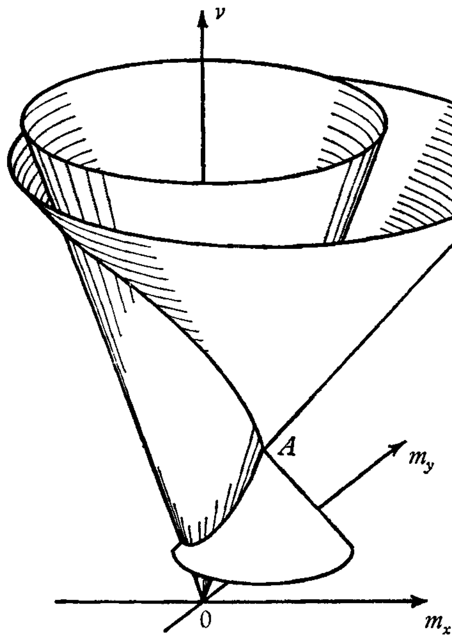

FIGURE 2. Schematic representation of the conditions for resonant interaction in the two-dimensional case. The external wave represented by A interacts with the external wave B where B can lie anywhere on either branch of the curve of intersection of the two cones.

When mh and mh' are sufficiently small, shallow-water theory is applicable and the results can be expressed analytically in a relatively simple way. Equation (2.1) reduces to

$$\rho\nu^4 - \rho\nu^2 g m^2(h+h') + (\rho-\rho')g^2 m^4 hh' = 0. \tag{2.2}$$

If the total depth of liquid is H and we put

$$h = \tfrac{1}{2}H(1+\alpha), \quad h' = \tfrac{1}{2}H(1-\alpha), \tag{2.3}$$

$$r^2 = \rho'/\rho < 1, \tag{2.4}$$

then
$$\left(\frac{\nu^2}{gHm^2}\right)^2 - \left(\frac{\nu^2}{gHm^2}\right) + \tfrac{1}{4}(1-r^2)(1-\alpha^2) = 0, \tag{2.5}$$

and
$$\frac{\nu^2}{m^2} = gH\left(\frac{1+a}{2}\right) = C_e^2 \quad \text{(external waves)}, \tag{2.6}$$

$$\frac{\nu^2}{m_2} = gH\left(\frac{1-a}{2}\right) = C_i^2 \quad \text{(internal waves)}, \tag{2.7}$$

where
$$a^2 = r^2 + \alpha^2 - r^2\alpha^2 \tag{2.8}$$

and C_e and C_i are the phase speeds of external and internal waves respectively. The quantity a^2 is a maximum, for any given value of r, when α is zero, i.e. when the layers are of equal depth.

Suppose there are two primary external waves $(\nu_1, \mathbf{m}_1)$ and $(\nu_2, \mathbf{m}_2)$ which both satisfy equation (2.5), so

$$\nu_1 = C_e m_1 \quad \text{and} \quad \nu_2 = C_e m_2. \tag{2.9}$$

We have assumed that the frequencies are necessarily positive, the direction of propagation being given by the direction of the vector wave-number $\mathbf{m}$. For resonance to occur one or other of the combination waves must satisfy the frequency equation for internal waves; either

$$\nu_1 + \nu_2 = C_i |\mathbf{m}_1 + \mathbf{m}_2|, \tag{2.10}$$

or

$$\nu_1 - \nu_2 = C_i |\mathbf{m}_1 - \mathbf{m}_2|. \tag{2.11}$$

From equation (2.9) we obtain

$$C_e^2(m_1 + m_2)^2 = C_i^2 |\mathbf{m}_1 + \mathbf{m}_2|^2, \tag{2.12}$$

or

$$C_e^2(m_1 - m_2)^2 = C_i^2 |\mathbf{m}_1 - \mathbf{m}_2|^2. \tag{2.13}$$

The former equation is never satisfied because $C_e > C_i$ and $m_1 + m_2 \geqslant |\mathbf{m}_1 + \mathbf{m}_2|$. The latter condition, which arises from the combination wave of wave-number $\mathbf{m}_3 = \mathbf{m}_1 - \mathbf{m}_2$, leads to the equation

$$S^2 - 2S\left(\frac{1 - b^2\cos\theta}{1 - b^2}\right) + 1 = 0, \tag{2.14}$$

where S is the ratio of the magnitudes of the vector wave-numbers ($S = m_2/m_1$), θ is the angle between them and

$$b^2 = C_i^2/C_e^2 = (1-a)/(1+a). \tag{2.15}$$

The locus of the vector wave-numbers $\mathbf{m}_2$ that resonate with a given wave-number $\mathbf{m}_1$ is sketched in figure 3.

When $\mathbf{m}_1$ and $\mathbf{m}_2$ have the same direction (i.e. $\theta = 0$) there is only the trivial solution, $\mathbf{m}_1 = \mathbf{m}_2$, $\mathbf{m}_3 = 0$. For all other values of θ there are two possible values of $\mathbf{m}_2$; in particular when the waves move in opposite directions (i.e. when $\theta = \pi$) we have:

$$\nu_2/\nu_1 = m_2/m_1 = (1-b)/(1+b) \quad \text{or} \quad (1+b)/(1-b), \tag{2.16}$$

$$m_3/m_1 = 2/(1+b) \quad \text{or} \quad 2/(1-b), \tag{2.17}$$

and

$$\nu_3/\nu_1 = 2b/(1+b) \quad \text{or} \quad 2b/(1-b). \tag{2.18}$$

To give a numerical example let us suppose that the layers are of equal depth and that $\rho'/\rho = 0{\cdot}9$. We then find that $b = 0{\cdot}16$ and

$$\nu_2/\nu_1 = m_2/m_1 = 0{\cdot}72,$$
$$m_3/m_1 = 1{\cdot}75,$$
$$\nu_3/\nu_1 = 0{\cdot}27.$$

As a second special case we assume that the upper layer is shallow and that the lower layer is deep, i.e. that mh is large and mh' is small, an assumption that is often appropriate for a range of wave numbers in the ocean. The frequency equation (2.1) now reduces to a simple form which has the two roots

$$\nu^2 = gm, \tag{2.19}$$

$$\nu^2 = gh'm^2(\rho-\rho')/\rho = c_i^2 m^2 \quad \text{say} \tag{2.20}$$

(see Lamb's *Hydrodynamics*, § 231), the former representing external waves, which behave as deep water waves, and the latter representing internal waves, which behave as shallow water waves. If we define ν_0 by

$$\nu_0^2 = g\rho/h'(\rho-\rho') = (g/c_i)^2, \tag{2.21}$$

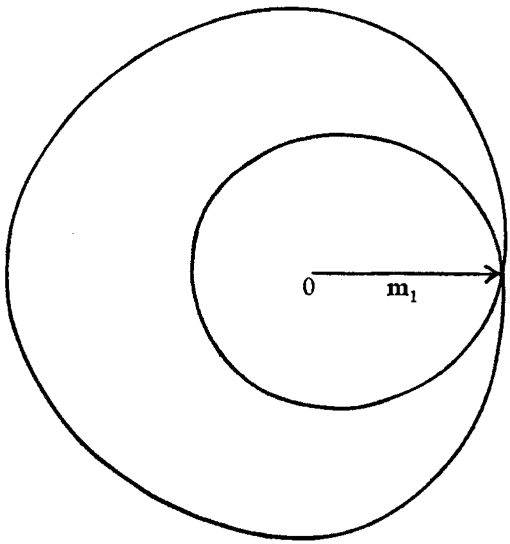

Figure 3. The locus of external vector wave-numbers that interact resonantly with the external wave $\mathbf{m}_1$.

and if ν is the frequency of an external wave, we have from (2.19) and (2.21)

$$(\nu/\nu_0)^2 = mh'(\rho-\rho')/\rho \ll 1, \tag{2.22}$$

since mh' is small by hypothesis and $(\rho-\rho')/\rho < 1$. Similarly,

$$(\nu/m)^2 = c_i^2\rho/mh'(\rho-\rho') \gg c_i^2. \tag{2.23}$$

If there are two external waves (ν_1, m_1) and (ν_2, m_2) then both satisfy equation (2.19) so $\nu_1^2 = gm_1$ and $\nu_2^2 = gm_2$. Resonance, which transfers energy to an internal wave, will occur if one or other of the combination waves satisfies the frequency equation (2.20) for internal waves, that is, if either equation (2.10) or (2.11) is satisfied. The former condition cannot be satisfied because $\nu_1+\nu_2 \gg c_i(m_1+m_2)$ (from (2.23)) and $m_1+m_2 \geqslant |\mathbf{m}_1+\mathbf{m}_2|$. The latter condition leads to

$$\nu_0^2(\nu_1-\nu_2)^2 = \nu_1^4+\nu_2^4-2\nu_1^2\nu_2^2\cos\theta, \tag{2.24}$$

where, as before, θ is the angle between the vector wave numbers $\mathbf{m}_1$ and $\mathbf{m}_2$. Let us put

$$\sigma = (\nu_1+\nu_2)/2\nu_0, \tag{2.25}$$

$$\epsilon = (\nu_1-\nu_2)/2\nu_0, \tag{2.26}$$

then by virtue of (2.22) σ and ϵ are both small and from (2.24) we obtain

$$\epsilon^2 = 4\epsilon^2\sigma^2 + (\sigma^2 - \epsilon^2)^2 \sin^2 \tfrac{1}{2}\theta, \tag{2.27}$$

whence

$$\epsilon^2 < (\sigma^2 + \epsilon^2)^2 \tag{2.28}$$

and ϵ must be of the order of σ^2.

Retaining only terms of the order of ϵ^2 in equation (2.27) we obtain the simple result

$$\epsilon = \sigma^2 \sin \tfrac{1}{2}\theta,$$

or

$$\nu_3 = \nu_1 - \nu_2 = (2\nu_0)^{-1}(\nu_1 + \nu_2)^2 \sin \tfrac{1}{2}\theta. \tag{2.29}$$

If $\theta = 0$ there is only the trivial solution $\nu_3 = 0$. In all other cases we can select two external waves of approximately the same frequency which will interact to transfer energy to internal motions. The difference in frequency is given by equation (2.29). As a numerical example let us suppose that the waves are moving in opposite directions ($\theta = \pi$), that $h' = 10^3$ cm and $(\rho - \rho')/\rho = 10^{-2}$. We then find that ν_0 is about 10 sec^{-1}. If we now consider external waves of frequency about 1 sec^{-1} which have wavelengths of about 60 m, then the difference in frequencies, from (2.29), is approximately 0·1 sec.

3. Detailed analysis of the shallow water case

Derivation of the equations

When the vertical acceleration is negligible by comparison with g, the relevant equations can be written

$$\partial \mathbf{V}'/\partial t + \mathbf{V}\,.\,\nabla \mathbf{V}' + g\nabla(h'\eta' + h\eta) = 0, \tag{3.1}$$

$$\partial \eta'/\partial t + \nabla\,.\,(\eta'\mathbf{V}') + \nabla\,.\,\mathbf{V}' = 0, \tag{3.2}$$

$$\partial \mathbf{V}/\partial t + \mathbf{V}\,.\,\nabla \mathbf{V} + g\nabla(r^2h'\eta' + h\eta) = 0, \tag{3.3}$$

$$\partial \eta/\partial t + \nabla\,.\,(\eta\mathbf{V}) + \nabla\,.\,\mathbf{V} = 0, \tag{3.4}$$

where, as before, the primed quantities refer to the upper layer and the unprimed quantities to the lower, h is the equilibrium depth, $h(1+\eta)$ is the actual depth (so that η is a dimensionless measure of the deviation from equilibrium) and $\mathbf{V}$ is the horizontal velocity. The method used in the following analysis is applicable whatever the value of h'/h; however, to avoid heavy algebra, attention is here confined to the case where the layers are of equal depth so α is zero in equations (2.3) and (2.8) and

$$h' = h = \tfrac{1}{2}H, \quad r = a. \tag{3.5}$$

The variables are first separated into 'external' and 'internal' variables defined by

$$\mathbf{V}_e = (1+r)(r\mathbf{V}' + \mathbf{V})/4r, \quad \mathbf{V}_i = (1-r)(r\mathbf{V}' - \mathbf{V})/4r, \tag{3.6}$$

$$\eta_e = (1+r)/r\eta' + \eta)/4r, \quad \eta_i = (1-r)(r\eta' - \eta)/4r, \tag{3.7}$$

and, as we shall see, a pure internal mode of motion is characterized by the vanishing of the external variables and vice versa. Using these variables as the dependent variables in equations (3.1)–(3.4) we find after appropriate transformation:

$$\partial \mathbf{V}_e/\partial t + \tfrac{1}{2}(1+r)\,gH\nabla\eta_e + \mathbf{V}_e.\nabla\mathbf{V}_e + \mathbf{V}_e.\nabla\mathbf{V}_i + \mathbf{V}_i.\nabla\mathbf{V}_e + b^{-4}\mathbf{V}_i.\nabla\mathbf{V}_i = 0, \quad (3.8)$$

$$\partial\eta_e/\partial t + \nabla.\mathbf{V}_e + \nabla.(\eta_e\mathbf{V}_e) + \nabla.(\eta_e\mathbf{V}_i + \eta_i\mathbf{V}_e + b^{-4}\eta_i\mathbf{V}_i) = 0, \quad (3.9)$$

$$\partial \mathbf{V}_i/\partial t + \tfrac{1}{2}(1-r)\,gH\nabla\eta_i + \mathbf{V}_i.\nabla\mathbf{V}_i + \mathbf{V}_i.\nabla\mathbf{V}_e + \mathbf{V}_e.\nabla\mathbf{V}_i + b^{4}\mathbf{V}_e.\nabla\mathbf{V}_e = 0, \quad (3.10)$$

$$\partial\eta_i/\partial t + \nabla.\mathbf{V}_i + \nabla.(\eta_i\mathbf{V}_i) + \nabla.(\eta_i\mathbf{V}_e + \eta_e\mathbf{V}_i + b^{4}\eta_e\mathbf{V}_e) = 0, \quad (3.11)$$

where b^2 is defined by equation (2.15) (with $a = r$). These equations are to be compared with those for a single layer:

$$\partial\mathbf{V}/\partial t + Hg\nabla\eta + \mathbf{V}.\nabla\mathbf{V} = 0, \quad (3.12)$$

and

$$\partial\eta/\partial t + \nabla.\mathbf{V} + \nabla.(\eta\mathbf{V}) = 0. \quad (3.13)$$

Equations (3.8)–(3.11) appear, at first sight, to be more complicated than those from which they were derived. It is possible, however, merely from the form of these equations, to say a great deal about the characteristics of internal and external modes of motion of a two-layer system. If the non-linear terms are negligible in equations (3.8)–(3.13) then only the first two terms remain in each equation. Equations (3.8)–(3.11) now form two *independent* pairs of equations, one pair for external motions and one pair for internal motions. Not only are these pairs isomorphic with one another but each pair is also isomorphic with the linearized equations for a single layer. The apparent gravity in the case of external motions is $(1+r)g/2$ which is approximately equal to (though always less than) g when the difference between the densities of the layers is small. The speed of external waves is then very close to the speed of waves in a single layer of depth H. On the other hand, the apparent gravity in the case of internal motions is $(1-r)g/2$ and this is much smaller than g when the density difference is small (i.e. when r is close to unity). Internal waves then travel much more slowly than external waves. Furthermore, in a pure external motion $\mathbf{V}_i$ is zero so the two layers move in phase with one another and the slip between the layers is small (though non-zero). In internal motion $\mathbf{V}_e$ is zero, the layers move in opposite phase and the slip between them is large; internal motions are therefore more rapidly dissipated by internal friction than are external motions of comparable energy.

Returning to equations (3.8)–(3.13) we notice that *the isomorphism extends in part to the non-linear terms*. The non-linear terms in equations (3.12) and (3.13) are associated with energy transfer between different scales and types of shallow water motion of a single layer. These terms occur in exactly the same form in each of the pairs of equations (3.8), (3.9) and (3.10), (3.11) (the transformation (3.6), (3.7) was in fact selected so that this would be so). The process of energy transfer between various scales and types of external motion (or between various scales and types of internal motion) is exactly the same as the process of transfer for a single layer. The other non-linear terms in equations (3.8)–(3.11) represent interactions and energy transfers between internal and external motions and have no counterparts in the equations (3.12), (3.13) for a single layer. It is this type of interaction with which we are concerned in the following analysis.

The equations (3.8)–(3.11) are applicable to all possible shallow water motions of the two-layer system, including motions involving rotation about a vertical axis as well as gravity waves. We exclude rotational motions from consideration by assuming that both curl $\mathbf{V}_e$ and curl $\mathbf{V}_i$ are zero. Furthermore, following Lorenz (1960), we make the 'maximum simplification' by considering only the non-linear terms that are relevant to the resonant interactions of a particular triplet comprising two external waves and one internal wave. The external waves are then modified by non-linear terms involving the products of internal and external variables and the internal waves are modified by terms involving products or squares of external variables. On these assumptions equations (3.8)–(3.11) simplify to:

$$\partial \mathbf{V}_e/\partial t + \tfrac{1}{2}(1+r)gH\nabla\eta_e + \nabla(\mathbf{V}_i.\mathbf{V}_e) = 0, \tag{3.14}$$

$$\partial \eta_e/\partial t + \nabla.\mathbf{V}_e + \nabla.(\eta_e\mathbf{V}_i + \eta_i\mathbf{V}_e) = 0, \tag{3.15}$$

$$\partial \mathbf{V}_i/\partial t + \tfrac{1}{2}(1-r)gH\nabla\eta_i + b^4\nabla(\tfrac{1}{2}\mathbf{V}_e^2) = 0, \tag{3.16}$$

$$\partial \eta_i/\partial t + \nabla.\mathbf{V}_i + b^4\nabla.(\eta_e\mathbf{V}_e) = 0. \tag{3.17}$$

We suppose that we have a solution of the form:

$$\eta_e = \eta_1 \sin(\nu_1 t - \mathbf{m}_1.\mathbf{r}) + \eta_2 \sin(\nu_2 t - \mathbf{m}_2.\mathbf{r}), \tag{3.18}$$

$$\mathbf{V}_e = C_e^2\left\{\frac{\eta_1\mathbf{m}_1}{\nu_1}\sin(\nu_1 t - \mathbf{m}_1.\mathbf{r}) + \frac{n_2\mathbf{m}_2}{\nu_2}\sin(\nu_2 t - \mathbf{m}_2.\mathbf{r})\right\}, \tag{3.19}$$

$$\eta_i = \eta_3 \sin(\nu_3 t - \mathbf{m}_3.\mathbf{r}), \tag{3.20}$$

$$\mathbf{V}_i = C_i^2\frac{\eta_3\mathbf{m}_3}{\nu_3}\sin(\nu_3 t - \mathbf{m}_3.\mathbf{r}), \tag{3.21}$$

where $\mathbf{m}_3 = \mathbf{m}_1 - \mathbf{m}_2$ and the resonance condition (2.13) is satisfied. We then use equation (1.9) to determine the rates of change of the three amplitudes η_1, η_2 and η_3.

Let us first consider the change in amplitude of the internal wave, η_3, caused by resonant interaction of the two external waves. We find, from equations (3.16) and (3.17),

$$\frac{\partial^2\eta_i}{\partial t_2} - C_i^2\nabla^2\eta_i = b^4\nabla.\left[\nabla\tfrac{1}{2}\mathbf{V}_e^2 - \frac{\partial}{\partial t}(\eta_e\mathbf{V}_e)\right]. \tag{3.22}$$

The terms on the right-hand side of equation (3.22) that are relevant in the resonant interaction are those involving $\cos(\nu_3 t - \mathbf{m}_3.\mathbf{r})$ which arise from the cross-product terms $\sin(\nu_1 t - \mathbf{m}_1.\mathbf{r})\sin(\nu_2 t - \mathbf{m}_2.\mathbf{r})$. Whence

$$\frac{\partial^2\eta_i}{\partial t^2} - C_i^2\nabla^2\eta_i = -\tfrac{1}{2}b^4\nu_3^2\eta_1\eta_2\left\{\frac{\mathbf{m}_1.\mathbf{m}_2}{m_1m_2} + b\left(\frac{\mathbf{m}_2.\mathbf{m}_3}{m_2m_3} + \frac{\mathbf{m}_3.\mathbf{m}_1}{m_3m_1}\right)\right\}\cos(\nu_3 t - \mathbf{m}_3.\mathbf{r}), \tag{3.23}$$

where, as in the introduction, we have put $\nu_3 = \nu_1 - \nu_2$ and $\mathbf{m}_3 = \mathbf{m}_1 - \mathbf{m}_2$ and we have used equations (2.9) and (2.11). Putting $\mathbf{m}_1.\mathbf{m}_2/m_1m_2$ equal to $\cos\theta$ and using the relation

$$\frac{\mathbf{m}_2.\mathbf{m}_3}{m_2m_3} + \frac{\mathbf{m}_3.\mathbf{m}_1}{m_3m_1} = b(1+\cos\theta), \tag{3.24}$$

(which is derived by putting $\mathbf{m}_3 = \mathbf{m}_1 - \mathbf{m}_2$ and using equation (2.13)) and equation (1.9) we find

$$d\eta_3/dt = \tfrac{1}{4}b^2\nu_3\eta_1\eta_2[\cos\theta + b^2(1+\cos\theta)]. \tag{3.25}$$

Similarly, the rates of change of η_1 and η_2 are given by

$$d\eta_1/dt = -\tfrac{1}{4}\nu_1\eta_2\eta_3[\cos\theta + b^2(1+\cos\theta)], \tag{3.26}$$

and

$$d\eta_2/dt = +\tfrac{1}{4}\nu_2\eta_3\eta_1[\cos\theta + b^2(1+\cos\theta)]. \tag{3.27}$$

Properties of the solutions

When other interactions are neglected, equations (3.25)–(3.27) completely describe the variation in the amplitudes of the three waves comprising a resonantly interacting triplet. If the factor $\cos\theta + b^2(1+\cos\theta)$ is zero then there is no primary interaction. Thus if b^2 is small (i.e. if the densities of the layers are nearly equal) then external wave trains, whose directions of propagation are at right angles to one another, will not interact even though the interaction conditions (2.13) are satisfied. On the other hand, if b^2 is close to unity (i.e. if the density of the upper layer is very much smaller than that of the lower) then external wave trains, whose directions of propagation are inclined at 120°, will not interact.

The first integrals of this set of equations are, of course, intimately related to the dynamical properties of the interacting triplet. Though there are only two such independent integrals these can appear in various guises; for instance the mean momentum of a liquid column of the two-layer system is given by

$$\mathbf{M} = \tfrac{1}{2}H(\rho'\overline{\eta'\mathbf{V}'} + \rho\overline{\eta\mathbf{V}}). \tag{3.28}$$

The bars represent time means over an interval that is long compared with the periods of the waves but sufficiently short for the changes in the amplitudes to be negligible. In terms of the internal and external variables (see equations (3.6) and (3.7)) we have

$$\mathbf{M} = \frac{4H\rho'}{(1-r)^2}(b^4\overline{\eta_e\mathbf{V}_e} + \overline{\eta_i\mathbf{V}_i}), \tag{3.29}$$

whence from equations (3.18)–(3.21)

$$\mathbf{M} = \frac{4\rho' HC_e^2}{1-r^2}\left\{b^2\left(\eta_1^2\frac{\mathbf{m}_1}{\nu_1} + \eta_2^2\frac{\mathbf{m}_2}{\nu_2}\right) + \eta_3^2\frac{\mathbf{m}_3}{\nu_3}\right\}. \tag{3.30}$$

It is immediately apparent from equations (3.25)–(3.27) and the definition of $\mathbf{m}_3$ that

$$\frac{d\mathbf{M}}{dt} = 0, \tag{3.31}$$

thus the momentum of the interacting triplet is constant. Similarly, the mean wave energy of a liquid column is

$$E = \tfrac{1}{4}H\{\rho'[\overline{V'^2} + \tfrac{1}{2}gH(\overline{\eta'^2} + 2\overline{\eta\eta'})] + \rho[\overline{V^2} + \tfrac{1}{2}gH\overline{\eta^2}]\}, \tag{3.32}$$

which transforms to

$$E = \frac{2\rho' g H^2}{1-r}\{b^2\overline{\eta_e^2}+\overline{\eta_i^2}\}$$

$$= \frac{4\rho' H C_e^2}{1-r^2}\{b^2(\eta_1^2+\eta_2^2)+\eta_3^2\}. \tag{3.33}$$

It is also apparent from equations (3.25)–(3.27) and the definition of ν_3 that

$$dE/dt = 0 \tag{3.34}$$

so that the wave energy of the interacting triplet is constant.

Equations (3.25)–(3.27) are of the form mentioned in the introduction where the rate of change of each amplitude is proportional to the product of the other

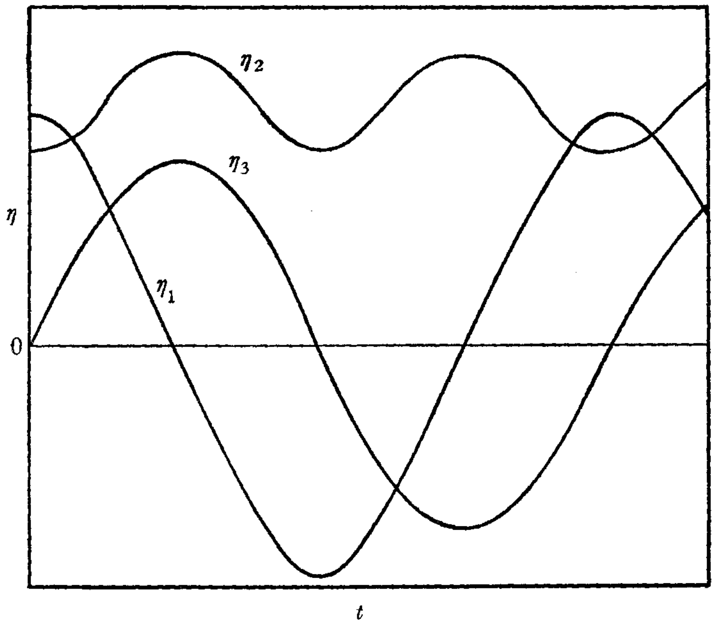

FIGURE 4. Variations in amplitudes of the waves comprising a resonantly interacting triplet.

two; the solutions are therefore Jacobian elliptic functions of time. We will briefly consider the case where the initial amplitude of the internal wave is zero and the initial amplitudes of the external waves are η_{10} and η_{20}. The appropriate solution is given by

$$\eta_1 = -\eta_{10}\,\mathrm{sn}\,(\gamma t - K), \tag{3.35}$$

$$\eta_2 = \eta_{20}\sec\beta\,\mathrm{dn}\,(\gamma t - K), \tag{3.36}$$

$$\eta_3 = \eta_{10}\, b(\nu_3/\nu_1)^{\frac{1}{2}}\,\mathrm{cn}\,(\gamma t - K), \tag{3.37}$$

where

$$\gamma = \tfrac{1}{2}b[\cos\theta + b(1+\cos\theta)]\,\eta_{10}(\nu_2\nu_3)^{\frac{1}{2}}\,\mathrm{cosec}\,\beta,$$

$$= \tfrac{1}{2}b[\cos\theta + b(1+\cos\theta)]\,\eta_{20}(\nu_1\nu_3)^{\frac{1}{2}}\sec\beta, \tag{3.38}$$

and the modulus of the elliptic functions is k where

$$k = \sin\beta, \tag{3.39}$$

$$\tan^2\beta = \left(\frac{\eta_{10}}{\eta_{20}}\right)^2 \frac{\nu_2}{\nu_1}, \tag{3.40}$$

and K is the complete elliptic integral

$$K = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}. \tag{3.41}$$

The period of oscillation of η_1 and η_3 is given by $4K/\gamma$ and the period of η_2 by $2K/\gamma$. The ratios of these periods to those of the waves themselves is about 10^3 and so we can justify *a posteriori* the assumption made in the derivation of equation (1.9).

The variations in amplitude are sketched in figure 4. The low-frequency external wave of amplitude η_2 never loses all its energy whereas the other external wave does lose all its energy after an interval $K/|\gamma|$, when the amplitude changes sign (i.e. there is a phase change of π). At this time the internal wave has its maximum amplitude and energy. In reality because of the rapid damping of the internal wave, it is unlikely that a complete oscillation of the type indicated in figure 4 could take place.

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